Stone-Dual Semantics for Natural Language

Introduction Mainstream possible-worlds semantics (PWS) follows Montague (1970) and Kripke (1963) in treating the set W of worlds as unstructured and the set P of propositions (meanings of utterances of declarative sentences) as its powerset. Here I advocate an alternative, topological, approach rooted in Stone's (1936, 1937) representation theory of boolean algebras and the related theory of boolean operators developed in the 1930s and 1940s by Tarski and others (e.g. Jónsson and Tarski 1951). On this approach W comes equipped with a topology T, and only certain sets of worlds—those which are **clopen** (simultaneously open and closed with respect to T) count as propositions. After sketching the basic formal apparatus of the proposed approach, I go on to show how it solves some foundational problems of mainstream PWS, and how it clarifies some key issues about modality, questions, and counterfactuals.

Formal Considerations Stone (1937) showed that there was a **duality** between boolean algebras and certain topological spaces, now called Stone spaces (compact Hausdorff spaces where the open sets are the unions of clopens, or equivalently, where the closed sets are the intersections of clopens). More specifically, Stone showed that any boolean algebra P is isomorphic to a boolean subalgebra of a powerset algebra $\wp(W)$, where W is the set of ultrafilters over P. The isomorphism, (the **Stone embedding**) maps each member of P to the set C_p whose members are the ultrafilters which have p as a member. But W forms a Stone space if the open sets are taken to be those subsets of W which are of the form $\bigcup S$, where each member of S is C_p for some $p \in P$. This Stone space is called the **spectrum** of P, written $\mathsf{Spec}(P)$. In fact, the clopens of $\mathsf{Spec}(P)$ are precisely the sets C_p (for $p \in P$). So the Stone embedding maps each $p \in P$ to the clopen C_p. Moreover, there is a bijection between the filters of P and the closed sets of W, that maps each filter F to the set F^* whose members are the ultrafilters with F as a subset.

Application to Natural Language Semantics Now let P be an infinite boolean algebra and $\langle W, T\rangle = \mathsf{Spec}(P)$. Think of W as the set of worlds, P (or the isomorphic lattice of clopens) as the set of propositions, and the order on P as the entailment relation on propositions. Think of the boolean connectives as the meanings of the NL 'logic words'. The filters over P can be thought of as partial worlds (situations). The Stone embedding from a proposition p to the corresponding clopen C_p can be thought of as mapping p to a Carnapian intension (identifying ultrafilters with Carnap's state descriptions), without requiring that the propositions actually *be* Carnapian intensions. It is crucially important that not just any old set of worlds corresponds to a proposition, only the clopen ones! On this approach, to say p is **true** at w is just to say that $w \in \mathsf{C}_p$, i.e. that w is one of the ultrafilters of which p is a member.

Tarski and others elaborated Stone's duality to include operators on Boolean algebras. Continuing to think of P as the propositions and W as the worlds, then according to this elaboration, unary operators on P can be thought of as propositional operators. In more current terminology, if $m \colon P \to P$, then m induces a **neighborhood system** (a function from worlds to sets of propositions) $N_m \colon W \to \wp(P)$ given by $p \in N_m(w)$ iff $m(p) \in w$. But m also induces a **relational frame** (function from worlds to sets of worlds) $R_m \colon W \to \wp(W)$ given by $w' \in R(w)$ iff $N_m(w) \subseteq w'$. It is not hard to see that there is a Galois connection between the set of neighborhood systems and the set of relational frames. This restricts to a bijection between the **filtered** neighborhood systems (ones where, for every w, $N(w)$ is a filter), and relational frames where the set of accessible sets at each world is a closed subset of $\mathsf{Spec}(P)$.

The remainder of the paper shows how adopting this perspective solves some foundational problems of standard PWS and clarifies some basic issues about modality, questions, and counterfactuals. These include: (1) the total omniscience problem arising from the existence of propositions which are singleton sets of worlds; (2) the status of nonprincipal ultrafilters over W; (3) the relationship between partition semantics vs. sets-of-answers semantics for questions; (4) the status of various notions of relative closeness (to a given world) of other worlds; and (5) the pragmatic factors at work in the interpretation of counterfactual conditionals.

References

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